BASES OF TOTAL WITT GROUPS AND LAX-SIMILITUDE

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ABSTRACT. We explain how to work with total Witt groups, more specifically, how to circumvent the classical embarrassment of making choices for line bundles up to isomorphisms and up to squares.

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INTRODUCTION

After completing our computation of the total Witt group $\mathrm{W}^{\mathrm{tot}}(\mathrm{Gr}(d,n))$ of Grassmannians in [3] and submitting that paper for publication, a somewhat rigid anonymous reader insisted on the problem that the total Witt group of a scheme Y

$$\mathrm{W}^{\mathrm{tot}}(Y) = \bigoplus_{[L]\in \mathrm{Pic}(Y)/2} \mathrm{W}^*(Y,L)$$

did not formally exist, because the group $\mathrm{W}^*(Y,L)$ depends on a choice of a representative L of $[L]\in\mathrm{Pic}(Y)/2$ up to *non-unique* isomorphism. Although formally correct, it might have occurred to a less rigid or less anonymous human being that this problem had nothing to do with Grassmannians per se, neither much to do with triangular Witt groups, and exists since the foundations of Witt groups of schemes themselves, starting with Knebusch [5].

For us, two alternatives presented themselves, beyond cowardly withdrawing the paper. First, we could trace all twists in use in the special case of Grassmannians, basically turning a reasonably short and hopefully readable paper about Witt groups into an obscure mess about line bundles. Or, alternatively, we could write another paper in which we *prove* that the choices do not really matter, for Grassmannians and much more generally, providing a tool for seven generations of Witt groupists to use happily ever after. For some reason, we went for the second alternative. The outcome is what the reader holds in her electronic hands.

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Convention. Throughout the paper, X and Y stand for regular noetherian separated schemes over $\mathbb{Z}[\frac{1}{2}]$, of finite Krull dimension. (See however Remark 7.4.)

For $i \in \mathbb{Z}$ and for a line bundle L over Y, the Witt group $\mathrm{W}^i_Z(Y,L)$ is the triangular Witt group of the derived category $\mathrm{D}^{\mathrm{b}}_Z(\mathrm{VB}(Y))$ of bounded complexes of vector bundles over Y with homology supported in a closed subset $Z \subset Y$, with respect to the usual duality derived from $\mathcal{H}om_{\mathcal{O}_Y}(-,L)$ and shifted i times. See [1, 2]. Of course, $\mathrm{W}^i(Y,L)$ stands for $\mathrm{W}^i_Y(Y,L)$. The line bundle L is often called *the twist* of the duality.

These Witt groups are periodic in two elementary ways. First they are 4-periodic in shift, i.e. there is an extremely natural isomorphism

$$\mathrm{W}^i_Z(Y,L) \cong \mathrm{W}^{i+4}_Z(Y,L)$$

by [1, Proposition 2.14]. On the other hand, we have a product

$$\mathrm{W}^{i_1}_{Z_1}(Y,L_1) \times \mathrm{W}^{i_2}_{Z_2}(Y,L_2) \to \mathrm{W}^{i_1+i_2}_{Z_1\cap Z_2}(Y,L_1\otimes L_2)$$

by [4] that recovers the usual multiplication on the classical Witt group $\mathrm{W}(Y) = \mathrm{W}^0(Y,\mathcal{O}_Y)$ and that turns any $\mathrm{W}^i_Z(Y,L)$ into a $\mathrm{W}(Y)$-module. For any line bundle M over Y, this product yields the other periodicity isomorphism, called the square-periodicity isomorphism,

$$\mathrm{W}^i_Z(Y,L) \xrightarrow{\sim} \mathrm{W}^i_Z(Y,M^{\otimes 2}\otimes L) \tag{1}$$

and given by multiplication with the Witt class $[M \overset{\sim}{\to} M^\vee \otimes M^{\otimes 2}] \in \mathrm{W}^0(Y,M^{\otimes 2})$, where $M^\vee$ denotes the dual of M. Finally, any isomorphism $L \overset{\sim}{\to} L'$ induces isomorphisms $\mathrm{W}^i_Z(Y,L) \xrightarrow{\sim} \mathrm{W}^i_Z(Y,L')$ in the obvious way.

From these isomorphisms, it is clear that all the Witt groups of a scheme can be recovered once we know the Witt groups $\mathrm{W}^i_Z(Y,L)$ for L and i varying in a set of representatives of $\mathrm{Pic}(Y)/2$ and $\mathbb{Z}/4$ respectively. The direct sum of such groups is what is usually called "the" total Witt group of the scheme Y (with support in Z). However, this total group is not canonical since it involves the choice of a line bundle L for every class in $\mathrm{Pic}(Y)/2$. Furthermore, if we want to turn this total Witt group into a ring, using the above product, we need to choose isomorphisms between $L_1 \otimes L_2$ and the chosen line bundle representing $[L_1 \otimes L_2]$ in $\mathrm{Pic}(Y)/2$, including the choice of "square roots" (for the periodicity modulo 2), and so on. All these choices should further satisfy some compatibilities, of the highest sex appeal. Last not least, it is unclear how to make such choices in a functorial way, when varying the scheme, under pull-back and under push-forward.

To circumvent such technical obstacles, we propose a way of keeping the intuitive simplification allowed by the total Witt group, yet avoiding the unpleasant use of a non-canonical object. Of course, it is unfortunately not true in full generality that one can completely ignore choices. Here, we provide a large class $\mathcal{S}_X$ of schemes Y over a given base X (Definition 4.1), in which it makes rigorous sense to say that a collection of Witt classes over Y form what we usually call a "basis of $\mathrm{W}^{\mathrm{tot}}(Y)$ over $\mathrm{W}^{\mathrm{tot}}(X)$".

The initial concept is that of *alignment* $A : L_1 \rightsquigarrow L_2$ between line bundles (Section 1) and the *alignment isomorphism*

$$A^{\circlearrowleft} \ : \ \mathrm{W}^*(Y,L_1) \overset{\sim}{\to} \mathrm{W}^*(Y,L_2)$$

induced on Witt groups (Section 2). We show how these interact with the two functorialities of Witt theory : pull-back and conditional push-forward. This leads us to introduce lax pull-backs and lax push-forwards (Section 3) which are heuristically pull-backs and push-forwards *in which one only cares about twists in* $\mathrm{Pic}/2$. This mathematical peace of mind is formally provided by Theorem 4.4, where we show

that a change of line bundle in some $\bar{Y}$ over Y can be descended to Y, as long as both $\bar{Y}$ and Y belong to our category $\mathcal{S}_X$.

In Sections 5 and 6, we generalize the action of "$\mathrm{W}^{\mathrm{tot}}(X)$" on "$\mathrm{W}^{\mathrm{tot}}(Y)$" for $Y \in \mathcal{S}_X$ by again allowing alignments to move classes around. In this context, we discuss the notion of "total basis".

We then show how the fundamental geometric results, localization long exact sequence, homotopy invariance and dévissage, behave with respect to the above flexibility (Section 7) and we explain how to trace a total basis of the total Witt group in such a localization sequence, *without* explicitly tracing the line bundles on the nose but only their classes in $\operatorname{Pic}/2$. See Theorem 7.1.

We have already made use of this formalism in the revised version of [3] and we hope that other people computing total Witt groups will enjoy the therapy.

1. The category of quadratic alignments

1.1. **Definition.** Let L_1 and L_2 be line bundles on a scheme Y. We say that a pair $A = (M, \phi)$ consisting of a line bundle M on Y and an isomorphism

$$\phi : M^{\otimes 2} \otimes L_1 \xrightarrow{\sim} L_2$$

is a *(quadratic) alignment* from L_1 to L_2. Such an alignment exists if and only if $[L_1] = [L_2]$ in $\operatorname{Pic}(Y)/2$. We use the following short notation for alignments:

$$A : L_1 \rightsquigarrow L_2 \,.$$

1.2. **Definition.** We say that two alignments $A = (M, \phi)$ and $A' = (M', \phi')$ from L_1 to L_2 are *isomorphic*, denoted $A \simeq A'$, if there exists an isomorphism $\tau : M \xrightarrow{\sim} M'$ such that the following diagram commutes:

$$\begin{tikzcd} M^{\otimes 2} \otimes L_1 \arrow[r, "\phi"] \arrow[d, "\tau^{\otimes 2} \otimes 1"'] & L_2 \arrow[d, equal] \\ M'^{\otimes 2} \otimes L_1 \arrow[r, "\phi'"] & L_2 \,. \end{tikzcd}$$

There is only a *set* of isomorphism classes of alignments from L_1 to L_2:

$$\mathrm{Al}_Y(L_1, L_2) := \{A : L_1 \rightsquigarrow L_2\}/\!\simeq .$$

We denote by $[A]_\simeq$ in $\mathrm{Al}(L_1, L_2)$ the isomorphism class of an alignment $A : L_1 \rightsquigarrow L_2$. We define the *alignment category*, denoted by

$$\mathcal{A}\ell(Y)\,,$$

to be the category of line bundles over Y with $\mathrm{Al}_Y(L_1, L_2)$ as morphism sets from L_1 to L_2. The composition of $L_1 \overset{A_1}{\rightsquigarrow} L_2 \overset{A_2}{\rightsquigarrow} L_3$ is defined as follows. If, say, $A_i = (M_i, \phi_i)$ with $\phi_i : M_i^{\otimes 2} \otimes L_i \xrightarrow{\sim} L_{i+1}$ for $i = 1, 2$, then

$$A_2 \circ A_1 := \big(M_2 \otimes M_1\,,\ \phi_3\big) \ :\ L_1 \rightsquigarrow L_3$$

where ϕ_3 is the obvious isomorphism (essentially $\phi_2 \circ \phi_1$)

$$(M_2 \otimes M_1)^{\otimes 2} \otimes L_1 \xrightarrow[(23)]{\sim} M_2^{\otimes 2} \otimes M_1^{\otimes 2} \otimes L_1 \xrightarrow[1 \otimes \phi_1]{\sim} M_2^{\otimes 2} \otimes L_2 \xrightarrow[\phi_2]{\sim} L_3 \,.$$

Composition is compatible with isomorphisms and is associative up to isomorphism hence turns $\mathcal{A}\ell(Y)$ into a category, in which id_L is given by $\big(\mathcal{O}_Y\,,\ \mathcal{O}_Y^{\otimes 2} \otimes L \cong L\big)$.

1.3. **Lemma.** *The category $\mathcal{A}\ell(Y)$ is a groupoid, i.e. for every alignment $A : L_1 \rightsquigarrow L_2$ there exists $A' : L_2 \rightsquigarrow L_1$ with $A \circ A' \simeq \mathrm{id}_{L_2}$ and $A' \circ A \simeq \mathrm{id}_{L_1}$, that is, $[A']_\simeq = [A]_\simeq^{-1}$. In particular, given A_1 and A_2 with same source (resp. same target) there always exists an alignment B such that $B \circ A_1 \simeq A_2$ (resp. $A_1 \circ B \simeq A_2$).*

Proof. If $A = (M, \phi)$ take $A' = (M^{-1}, \phi') : L_2 \rightsquigarrow L_1$ where ϕ' is $(M^{-1})^{\otimes 2} \otimes \phi^{-1}$ followed by the canonical isomorphism $(M^{-1})^{\otimes 2} \otimes M^{\otimes 2} \otimes L_1 \cong L_1$. □

1.4. **Definition.** We can also tensor alignments: $(L_1 \overset{A_1}{\rightsquigarrow} L_1') \otimes (L_2 \overset{A_2}{\rightsquigarrow} L_2')$. If, say, $A_i = (M_i, \phi_i)$ with $\phi_i : M_i^{\otimes 2} \otimes L_i \xrightarrow{\sim} L_i'$ for $i = 1, 2$, then their *tensor product*

$$A_1 \otimes A_2 := \big(M_1 \otimes M_2\,, \phi_4\big) \;:\; L_1 \otimes L_2 \rightsquigarrow L_1' \otimes L_2'$$

is given by the obvious isomorphism (ϕ_4 is morally essentially $\phi_1 \otimes \phi_2$)

$$(M_1 \otimes M_2)^{\otimes 2} \otimes L_1 \otimes L_2 \xrightarrow[(2453)]{\cong} {M_1}^{\otimes 2} \otimes L_1 \otimes {M_2}^{\otimes 2} \otimes L_2 \xrightarrow[\phi_1 \otimes \phi_2]{\sim} L_1' \otimes L_2' \,.$$

The reader can verify that this tensor product preserves isomorphisms of alignments and turns $\mathcal{A}\ell(Y)$ into a symmetric monoidal category.

1.5. **Lemma.** *Every line bundle L over Y is invertible in $\mathcal{A}\ell(Y)$ for this $\otimes$. In particular, the map $L \otimes -$ induces a bijection $\mathrm{Al}_Y(L_1, L_2) \xrightarrow{\sim} \mathrm{Al}_Y(L \otimes L_1\,,\, L \otimes L_2)$ and similarly for $- \otimes L$.*

Proof. Indeed, L^{-1} is also the inverse of L in $\mathcal{A}\ell(Y)$. □

Finally, the following remark contains some functoriality of alignments with respect to the scheme Y:

1.6. **Remark.** Given a morphism $f : \bar{Y} \to Y$ and an alignment $A = (M, \phi) : L_1 \rightsquigarrow L_2$ on Y, there is an obvious alignment $f^*(A) := (f^*M, f^*\phi) : f^*L_1 \rightsquigarrow f^*L_2$. The reader will verify functoriality of this construction: $A \simeq A' \implies f^*A \simeq f^*A'$ and $f^*(A_2 \circ A_1) \simeq f^*(A_2) \circ f^*(A_1)$ and $f^*(A_1 \otimes A_2) \simeq f^*A_1 \otimes f^*A_2$ and $(gf)^*(A) \simeq f^*(g^*(A))$. So we get a well-defined $\otimes$-functor

$$f^* : \mathcal{A}\ell(Y) \longrightarrow \mathcal{A}\ell(\bar{Y})$$

such that $(gf)^* \simeq f^* g^*$.

If Y and $\bar{Y}$ are regular and $f : \bar{Y} \to Y$ is proper, with relative canonical bundle ω_f, and if $A = (M, \phi) :\ L_1 \rightsquigarrow L_2$ is an alignment on Y, we define an alignment $f^!(A) :\ \omega_f \otimes f^*L_1 \rightsquigarrow \omega_f \otimes f^*L_2$ on $\bar{Y}$ by

$$f^!(A) = \big(f^*M\,,\, f^!\phi_2\big) \tag{2}$$

where $f^!\phi_2$ is the canonical isomorphism

$$f^*M^{\otimes 2} \otimes \omega_f \otimes f^*L_1 \xrightarrow[(123)]{\sim} \omega_f \otimes f^*M^{\otimes 2} \otimes f^*L_1 \cong \omega_f \otimes f^*\big(M^{\otimes 2} \otimes L_1\big) \xrightarrow[1 \otimes f^*\phi]{\sim} \omega_f \otimes f^*L_2 \,.$$

Using the monoidal structure, this can be stated as $f^!(A) = \mathrm{id}_{\omega_f} \otimes f^*(A)$. In particular, $f^!$ is as functorial as f^* was, except that $f^!$ is not monoidal.

2. Alignment isomorphisms on Witt groups

2.1. **Notation.** Let $\phi : L \xrightarrow{\sim} L'$ be an isomorphism of line bundles on a scheme Y. The isomorphism induced on Witt groups is denoted by

$$\underline{\phi} : \mathrm{W}^i_Z(Y, L) \xrightarrow{\sim} \mathrm{W}^i_Z(Y, L') \,.$$

The square periodicity isomorphism associated to a line bundle M is denoted by

$$\mathrm{per}_M \;: \mathrm{W}^i_Z(Y, L) \xrightarrow{\sim} \mathrm{W}^i_Z(Y, M^{\otimes 2} \otimes L).$$

2.2. **Remark.** Here are some easy compatibilities between those isomorphisms, that we leave to the reader. We use the obvious notation.

(i) $\underline{\phi'} \circ \underline{\phi} = \underline{\phi' \circ \phi}$.

(ii) $\mathrm{per}_{M_2} \circ \mathrm{per}_{M_1} = \underline{(23)} \circ \mathrm{per}_{M_2 \otimes M_1}$ for $(M_2 \otimes M_1)^{\otimes 2} \otimes L \underset{(23)}{\overset{\cong}{\to}} M_2^{\otimes 2} \otimes M_1^{\otimes 2} \otimes L$.

(iii) $\mathrm{per}_M \circ \underline{\phi} = \underline{(1 \otimes \phi)} \circ \mathrm{per}_M$ where $1 \otimes \phi : M^{\otimes 2} \otimes L_1 \overset{\sim}{\to} M^{\otimes 2} \otimes L_2$.

(iv) $\mathrm{per}_{M'} = \underline{(\tau^{\otimes 2} \otimes 1)} \circ \mathrm{per}_M$, for every isomorphism $\tau : M \overset{\sim}{\to} M'$.

It follows that any composition of per_{M_i} and $\underline{\phi_j}$ (in any order) can always be reduced to one composition of the form $\underline{\phi} \circ \mathrm{per}_M$. This is the true reason for the notion of alignment introduced in Section 1 and yields naturally:

2.3. **Definition.** For every alignment $A = (M, \phi) : L_1 \rightsquigarrow L_2$ (Definition 1.1), i.e. $\phi : M^{\otimes 2} \otimes L_1 \overset{\sim}{\to} L_2$, we define an isomorphism $A^{\circlearrowleft} := \underline{\phi} \circ \mathrm{per}_M$

$$A^{\circlearrowleft} \;:\; \mathrm{W}^*_Z(Y, L_1) \xrightarrow{\sim} \mathrm{W}^*_Z(Y, L_2)$$

that we call the *alignment isomorphism (on Witt groups)* associated to the alignment $A : L_1 \rightsquigarrow L_2$.

2.4. **Example.** To a unit $u \in \mathcal{O}_Y(Y)^\times$ we can associate two things: An isomorphism $u\cdot : L \overset{\sim}{\to} L$ for any line bundle L and a Witt class $\langle u \rangle$ in $\mathrm{W}(Y) = \mathrm{W}^0(Y, \mathcal{O}_Y)$. It is an easy exercise to verify that multiplication by $\langle u \rangle$ on Witt groups is the same as the alignment isomorphism $\underline{(u\cdot)} = A_u{}^{\circlearrowleft}$, where $A_u = \big(\mathcal{O}_Y, (u\cdot)\big)$.

With this example in mind, it seems reasonable to use the following terminology:

2.5. **Definition.** Let us say that two Witt classes $w_1 \in \mathrm{W}^j_Z(Y, L_1)$ and $w_2 \in \mathrm{W}^j_Z(Y, L_2)$ are *lax-similar* if there exists an alignment $A : L_1 \rightsquigarrow L_2$ such that $A^{\circlearrowleft}(w_1) = w_2$. This is an equivalence relation, that we denote by

$$w_1 \leftrightsquigarrow w_2 \,.$$

2.6. **Remark.** We have $w \leftrightsquigarrow 0$ if and only if $w = 0$. However, Witt classes up to lax-similitude do not form a group, as we already know from ordinary similitude.

2.7. **Proposition.** *The assignment $A \mapsto A^{\circlearrowleft}$ respects the structures of Section 1:*

(a) Isomorphic alignments $A \simeq A'$ induce the same *alignment isomorphism $A^{\circlearrowleft} = A'^{\circlearrowleft}$ on Witt groups. Hence $(-)^{\circlearrowleft}$ is well-defined on $\mathcal{A}\ell(Y)$.*

(b) Given two alignments $L_1 \overset{A_1}{\rightsquigarrow} L_2 \overset{A_2}{\rightsquigarrow} L_3$, we have

$$(A_2 \circ A_1)^{\circlearrowleft} = A_2{}^{\circlearrowleft} \circ A_1{}^{\circlearrowleft}$$

on Witt groups. Hence $(-)^{\circlearrowleft}$ is functorial.

(c) Given two alignments $A_i : L_i \rightsquigarrow L_i'$ and two Witt classes $w_i \in \mathrm{W}^{j_i}_{Z_i}(Y, L_i)$, for $i = 1, 2$, we have

$$(A_1 \otimes A_2)^{\circlearrowleft}(w_1 \cdot w_2) = \big(A_1{}^{\circlearrowleft}(w_1)\big) \cdot \big(A_2{}^{\circlearrowleft}(w_2)\big)\,.$$

in $\mathrm{W}^{j_1+j_2}_{Z_1 \cap Z_2}(Y, L_1' \otimes L_2')$. Hence $(-)^{\circlearrowleft}$ is (somewhat) monoidal.

Proof. Part (a) follows from Remark 2.2 (i) and (iv). For (b), contemplate

$$\begin{tikzcd}
\mathrm{W}^*_Z(Y, L_1) \arrow[r, "\mathrm{per}_{M_1}"] \arrow[d, "\mathrm{per}_{M_2 \otimes M_1}"] & \mathrm{W}^*_Z(Y, M_1^{\otimes 2} \otimes L_1) \arrow[r, "\underline{\phi_1}"] \arrow[d, "\mathrm{per}_{M_2}"] & \mathrm{W}^*_Z(Y, L_2) \arrow[d, "\mathrm{per}_{M_2}"] \\
\mathrm{W}^*_Z(Y, (M_2 \otimes M_1)^{\otimes 2} \otimes L_1) \arrow[r, "\underline{(23)}"'] & \mathrm{W}^*_Z(Y, M_2^{\otimes 2} \otimes M_1^{\otimes 2} \otimes L_1) \arrow[r, "\underline{1 \otimes \phi_1}"'] & \mathrm{W}^*_Z(Y, M_2^{\otimes 2} \otimes L_2)
\end{tikzcd}$$

This diagram commutes by Remark 2.2 (ii) and (iii). Post-composing the two outside compositions with $\underline{\phi_2} : \mathrm{W}^*_Z(Y, M_2{}^{\otimes 2} \otimes L_2) \overset{\sim}{\to} \mathrm{W}^*_Z(Y, L_3)$ yields the result. Part (c) is straightforward from the definition of the Witt group product [4]. $\square$

2.8. **Remark.** Since $\mathcal{A}\ell(Y)$ is a groupoid (Lemma 1.3), Proposition 2.7 (b) gives us that any zig-zag of alignment isomorphisms (on a given scheme) can be realized by one single alignment isomorphism. Note however that there might be non-isomorphic alignments from L_1 to L_2, hence possibly non-equal alignment isomorphisms on Witt groups.

Compatibility of $A^{\circlearrowleft}$ with morphisms $f : \bar{Y} \to Y$ is the subject of Section 3.

2.9. **Remark.** It is easy to verify that for every alignment $A : L_1 \rightsquigarrow L_2$ the alignment isomorphism $A^{\circlearrowleft} : \mathrm{W}^*_Z(Y, L_1) \xrightarrow{\sim} \mathrm{W}^*_Z(Y, L_2)$ commutes with the periodicity isomorphism $\mathrm{W}^* \xrightarrow{\sim} \mathrm{W}^{*+4}$, the latter being simply induced on the underlying triangulated category by the square of the shift Σ^2. This automorphism Σ^2 commutes with all operations involved in $A^{\circlearrowleft}$. Note that Σ^2 has no hidden sign either.

2.10. **Remark.** Alignment isomorphisms are compatible with localization long exact sequences. Compatibility of $A^{\circlearrowleft}$ with restriction to an open subscheme is a special case of Corollary 3.2 below. Let us discuss the other two types of homomorphisms.

(1) Let $Z \subset Z'$ be closed subsets of Y. Then extension-of-support $\mathrm{e} : \mathrm{W}^*_Z(Y, L) \to \mathrm{W}^*_{Z'}(Y, L)$ obviously commutes with alignment isomorphism $A^{\circlearrowleft}$.

(2) For the connecting homomorphism, let $U = Y \smallsetminus Z$ and consider $A : L \rightsquigarrow L'$ over Y and its obvious restriction $A_{|_U} : L_{|_U} \rightsquigarrow L'_{|_U}$. Then the following diagram

$$\begin{tikzcd}
\mathrm{W}^j(U, L_{|_U}) \arrow[r, "\partial"] \arrow[d, "{A_{|_U}}^{\circlearrowleft}"'] & \mathrm{W}^{j+1}_Z(Y, L) \arrow[d, "A^{\circlearrowleft}"] \\
\mathrm{W}^j(U, L'_{|_U}) \arrow[r, "\partial"] & \mathrm{W}^{j+1}_Z(Y, L')
\end{tikzcd}$$

commutes. Indeed, if $A = (M, \phi)$, then $A^{\circlearrowleft} = \underline{\phi} \circ \mathrm{per}_M$ and ∂ clearly commutes with $\underline{\phi}$. It is proved in [4] that ∂ is $\mathrm{W}^*(Y)$-linear, which explains why it commutes with per_M.

3. Lax pull-back and lax push-forward

We assume for simplicity (in the treatment of the push-forward) that all schemes Y, $\bar{Y}$, etc., are smooth over our regular base X.

Given a morphism $f : \bar{Y} \to Y$ and a closed subset Z of Y, we have a pull-back

$$f^* : \mathrm{W}^*_Z(Y, L) \to \mathrm{W}^*_{f^{-1}(Z)}(\bar{Y}, f^*L)\,.$$

On the other hand, when f is proper and has constant relative dimension $\dim(f)$ (the latter is automatic if $\bar{Y}$ is connected) and has relative canonical bundle ω_f, we have a push-forward homomorphism for every closed subset $\bar{Z} \subset \bar{Y}$

$$f_* : \mathrm{W}^{*+\dim(f)}_{\bar{Z}}\big(\bar{Y}, \omega_f \otimes f^*L\big) \longrightarrow \mathrm{W}^*_{f(\bar{Z})}(Y, L)\,.$$

3.1. **Remark.** Continuing Remark 2.2, for every morphism of schemes $f : \bar{Y} \to Y$, we have naturality:

(v) $f^* \circ \underline{\phi} = \underline{f^*\phi} \circ f^*$.

(vi) $f^* \circ \mathrm{per}_M = \mathrm{per}_{f^*M} \circ f^*$.

Finally, if $f : \bar{Y} \to Y$ is proper with relative canonical bundle ω_f, we have:

(vii) $\underline{\phi} \circ f_* = f_* \circ \big(\underline{1 \otimes f^*\phi}\big)$ for the canonical $1 \otimes f^*\phi : \omega_f \otimes f^*L_1 \xrightarrow{\sim} \omega_f \otimes f^*L_2$.

(viii) $\mathrm{per}_M \circ f_* = f_* \circ \underline{(123)} \circ \mathrm{per}_{f^*M}$ by the projection formula, where $(123) : (f^*M)^{\otimes 2} \otimes \omega_f \otimes f^*L_1 \xrightarrow{\cong} \omega_f \otimes f^*\big(M^{\otimes 2} \otimes L_1\big)$ is the canonical isomorphism.

3.2. **Corollary.** *Recall Remark 1.6 for f^* and $f^!$ on alignments. For $f:\bar{Y}\to Y$ and for an alignment $A=(M,\phi): L_1\rightsquigarrow L_2$ on Y, we have on Witt groups*

$$f^*\circ A^{\circlearrowleft}=(f^*A)^{\circlearrowleft}\circ f^*\,.$$

When f is proper, we have on Witt groups

$$A^{\circlearrowleft}\circ f_*=f_*\circ(f^!A)^{\circlearrowleft}\,.$$

Proof. This is a compact form of Remark 3.1. □

3.3. **Remark.** Given a morphism $f:\bar{Y}\to Y$, we can compose the pull-back f^* with alignment isomorphisms on Y and on $\bar{Y}$. By Corollary 3.2, any such composition can be brought down to one of the form $\bar{A}^{\circlearrowleft}\circ f^*$ for some alignment $\bar{A}$ on $\bar{Y}$. Similarly, one can combine the push-forward f_* with alignment isomorphisms on Y and $\bar{Y}$ and again the alignment isomorphisms on Y are redundant, i.e. such a composition always boils down to one of the form $f_*\circ\bar{A}^{\circlearrowleft}$ for some alignment $\bar{A}$ on $\bar{Y}$. Let us give names to those generalized pull-back and push-forward.

3.4. **Definition.** Let $f:\bar{Y}\to Y$ be a morphism. Let L and $\bar{L}$ be line bundles on Y and $\bar{Y}$ respectively and let $\bar{A}: f^*L\rightsquigarrow\bar{L}$ be an alignment on $\bar{Y}$ (Definition 1.1). We define the *lax pull-back homomorphism* from $\mathrm{W}^*_Z(Y,L)$ to $\mathrm{W}^*_{f^{-1}(Z)}(\bar{Y},\bar{L})$ by

$$\mathrm{Pull}_{\bar{A},f}:=\bar{A}^{\circlearrowleft}\circ f^*\;:\quad \mathrm{W}^*_Z(Y,L)\longrightarrow \mathrm{W}^*_{f^{-1}(Z)}(\bar{Y},f^*L)\xrightarrow{\sim}\mathrm{W}^*_{f^{-1}(Z)}(\bar{Y},\bar{L})\,.$$

3.5. **Definition.** Let $f:\bar{Y}\to Y$ be a proper morphism with relative canonical bundle ω_f and relative dimension $d=\dim(f)$. Let L and $\bar{L}$ be line bundles on Y and $\bar{Y}$ respectively, and let $\bar{A}:\bar{L}\rightsquigarrow\omega_f\otimes f^*L$ be an alignment on $\bar{Y}$ (note ω_f here). We define the *lax push-forward homomorphism* from $\mathrm{W}^{*+d}_{\bar{Z}}(\bar{Y},\bar{L})$ to $\mathrm{W}^*_{f(\bar{Z})}(Y,L)$ by

$$\mathrm{Push}_{f,\bar{A}}:=f_*\circ\bar{A}^{\circlearrowleft}\;:\quad \mathrm{W}^{*+d}_{\bar{Z}}(\bar{Y},\bar{L})\xrightarrow{\sim}\mathrm{W}^{*+d}_{\bar{Z}}(\bar{Y},\omega_f\otimes f^*L)\longrightarrow\mathrm{W}^*_{f(\bar{Z})}(Y,L)\,.$$

By Proposition 2.7 (a), both $\mathrm{Pull}_{\bar{A},f}$ and $\mathrm{Push}_{f,\bar{A}}$ only depend on the isomorphism class of $\bar{A}$.

3.6. **Proposition.** *For composable $\tilde{Y}\xrightarrow{g}\bar{Y}\xrightarrow{f}Y$ and for alignments $\bar{A}$ on $\bar{Y}$ and $\tilde{A}$ on $\tilde{Y}$ such that $\tilde{A}$ and $g^*\bar{A}$ are composable, we have*

$$\mathrm{Pull}_{\tilde{A},\,g}\circ\mathrm{Pull}_{\bar{A},\,f}=\mathrm{Pull}_{\tilde{A}\circ(g^*\bar{A})\,,\,fg}\,.$$

If instead $g^!\bar{A}$ and $\tilde{A}$ are composable and if f and g are moreover proper, then

$$\mathrm{Push}_{f,\,\bar{A}}\circ\mathrm{Push}_{g,\,\tilde{A}}=\mathrm{Push}_{fg\,,\,(g^!\bar{A})\circ\tilde{A}}$$

Proof. Direct by Corollary 3.2, Definitions 3.4 and 3.5 and Proposition 2.7 (b). □

4. Descending alignments

Given a morphism $f:\bar{Y}\to Y$, it might happen that two line bundles L_1 and L_2 on Y have f^*L_1 and f^*L_2 aligned over $\bar{Y}$, in the sense of Definition 1.1, without necessarily being aligned over Y. This can cause sorrow in the taverns. We propose a simple solution, which will be convenient in applications.

4.1. **Definition.** Recall our separated, noetherian regular base scheme X over $\mathbb{Z}[\frac{1}{2}]$. Let Y be a scheme over X. We denote by $\pi_Y:Y\to X$ the structure morphism and by $\mathrm{Pic}_X(Y)$ the cokernel of $\pi_Y^*:\mathrm{Pic}(X)\to\mathrm{Pic}(Y)$. Consider the full subcategory

$$\mathcal{S}_X$$

of smooth X-schemes $\pi_Y:Y\to X$ (with morphisms over X of course) such that:

(I) The map $\pi_Y^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is injective.
(II) The abelian group $\mathrm{Pic}_X(Y) = \mathrm{Pic}(Y)/\pi_Y^*\big(\mathrm{Pic}(X)\big)$ has no 2-torsion.
(III) The map $\pi_Y^* : \mathbb{G}_\mathrm{m}(X) \to \mathbb{G}_\mathrm{m}(Y)/\mathbb{G}_\mathrm{m}(Y)^2$ is surjective.

4.2. **Remark.** Note that X itself is in $\mathcal{S}_X$. Projective spaces over X, Grassmannians and various flag varieties over X are in $\mathcal{S}_X$, as explained in [3]. If X is local (e.g. the spectrum of a field) and thus has trivial Picard group, then projective varieties over X with no 2-torsion in their Picard group are in $\mathcal{S}_X$.

Assumptions (I) and (II) allow an easy chase:

4.3. **Lemma.** *Let $f : \bar{Y} \to Y$ be a morphism in $\mathcal{S}_X$. Then:*

(a) The homomorphism $f^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(\bar{Y})$ induces an isomorphism on 2-torsion subgroups ${}_2\mathrm{Pic}(Y) \xrightarrow{\sim} {}_2\mathrm{Pic}(\bar{Y})$.
(b) The sequence $0 \to \mathrm{Pic}(X)/2 \to \mathrm{Pic}(Y)/2 \to \mathrm{Pic}_X(Y)/2 \to 0$ is exact.
(c) The homomorphism $\mathrm{Pic}(Y)/2 \longrightarrow \mathrm{Pic}_X(Y)/2 \oplus \mathrm{Pic}(\bar{Y})/2$ is injective.
(d) If a line bundle $\bar{L}$ on $\bar{Y}$ is such that $[\bar{L}] = [f^*L]$ in $\mathrm{Pic}_X(\bar{Y})/2$ for some L over Y, then there exists L' over Y with $[L'] = [L]$ in $\mathrm{Pic}_X(Y)/2$ and $[\bar{L}] = [f^*L']$ in $\mathrm{Pic}(\bar{Y})/2$ already.

Proof. Since $\pi_Y f = \pi_{\bar{Y}}$ it suffices to prove (a) for $f = \pi_Y : Y \to X$. Multiplication by 2 yields an endomorphism of the following short sequence of abelian groups

$$0 \to \mathrm{Pic}(X) \xrightarrow{\pi_Y^*} \mathrm{Pic}(Y) \longrightarrow \mathrm{Pic}_X(Y) \to 0,$$

which is exact by (I) above. The Snake Lemma and Assumption (II) give (a) and (b). For (c), it suffices to compare the exact sequences (b) for Y and $\bar{Y}$ via f^*:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(X)/2 \arrow[r] \arrow[d, equal] & \mathrm{Pic}(Y)/2 \arrow[r] \arrow[d, "f^*"] & \mathrm{Pic}_X(Y)/2 \arrow[r] \arrow[d, "f^*"] & 0 \\ 0 \arrow[r] & \mathrm{Pic}(X)/2 \arrow[r] & \mathrm{Pic}(\bar{Y})/2 \arrow[r] & \mathrm{Pic}_X(\bar{Y})/2 \arrow[r] & 0 \end{tikzcd}$$

to chase the wanted injectivity (in fact, the right-hand square is cartesian). For (d), if there exists a line bundle K on X with $[\bar{L}] = [f^*L \otimes \pi_{\bar{Y}}^* K] = [f^*(L \otimes \pi_Y^* K)]$ in $\mathrm{Pic}(\bar{Y})/2$, take $L' := L \otimes \pi_Y^* K$, which is in the same class as L in $\mathrm{Pic}_X(Y)/2$. $\square$

Here is the announced descent of alignments along morphisms $\bar{Y} \to Y$ in $\mathcal{S}_X$.

4.4. **Theorem.** *Let $f : \bar{Y} \to Y$ be a morphism in $\mathcal{S}_X$. Let L_1 and L_2 be line bundles on Y and let $\bar{A} : f^*L_1 \rightsquigarrow f^*L_2$ be an alignment on $\bar{Y}$ (Definition 1.1). Suppose that $[L_1] = [L_2]$ in $\mathrm{Pic}_X(Y)/2$. Then there exists an alignment $A : L_1 \rightsquigarrow L_2$ on Y and an isomorphism $f^*A \simeq \bar{A}$ on $\bar{Y}$ (see Definition 1.2 and 1.6).*

Proof. Write $\bar{A} = (\bar{M}, \bar{\phi})$ so that $\bar{\phi}$ is an isomorphism $\bar{M}^{\otimes 2} \otimes f^*L_1 \xrightarrow{\sim} f^*L_2$ on $\bar{Y}$. In particular, in the group $\mathrm{Pic}(\bar{Y})/2$, we have $[f^*L_1] = [f^*L_2]$. Since we also assume $[L_1] = [L_2]$ in $\mathrm{Pic}_X(Y)/2$, we get by Lemma 4.3 (c) that $[L_1] = [L_2]$ in $\mathrm{Pic}(Y)/2$. Hence there exists a line bundle M' over Y such that $M'^{\otimes 2} \otimes L_1 \simeq L_2$. Pulling-back to $\bar{Y}$, we get $(f^*M')^{\otimes 2} \simeq \bar{M}^{\otimes 2}$ over $\bar{Y}$. Hence $[(f^*M') \otimes \bar{M}^{-1}] \in {}_2\mathrm{Pic}(\bar{Y}) \xleftarrow{\sim} {}_2\mathrm{Pic}(Y)$ by Lemma 4.3 (a). Hence there exists $[M''] \in {}_2\mathrm{Pic}(Y)$ such that $f^*(M' \otimes M'') \simeq \bar{M}$. Defining $M := M' \otimes M''$ we now have

$$f^*M \simeq \bar{M} \qquad \text{and also} \qquad M^{\otimes 2} \otimes L_1 \simeq L_2$$

since $M'^{\otimes 2} \otimes L_1 \simeq L_2$ already and $M''^{\otimes 2}$ is trivial.

Now choose two isomorphisms $\bar{\tau}_0 : f^*M \overset{\sim}{\to} \bar{M}$ and $\phi_0 : M^{\otimes 2} \otimes L_1 \overset{\sim}{\to} L_2$, which exist by the above construction, and consider the following diagram over $\bar{Y}$

$$
\begin{tikzcd}
(f^*M)^{\otimes 2} \otimes f^*L_1 \arrow[r, "\bar{\tau}_0^{\otimes 2} \otimes 1"] \arrow[d, equal] & \bar{M}^{\otimes 2} \otimes f^*L_1 \arrow[d, "\bar{\phi}"] \\
f^*(M^{\otimes 2} \otimes L_1) \arrow[r, "f^*\phi_0"] & f^*L_2 .
\end{tikzcd}
\tag{3}
$$

A priori, this only commutes up to a unit $t \in \mathbb{G}_{\mathrm{m}}(\bar{Y})$, like every diagram of isomorphisms of line bundles. By Assumption (III), $t = \pi_{\bar{Y}}^*(u) \cdot v^2$ for some $u \in \mathbb{G}_{\mathrm{m}}(X)$ and $v \in \mathbb{G}_{\mathrm{m}}(\bar{Y})$. Hence we can define $\phi = \pi_Y^*(u) \cdot \phi_0$ and $\bar{\tau} = v^{-1} \cdot \bar{\tau}_0$, so that ϕ and $\bar{\tau}$ make Diagram (3) strictly commute (in place of ϕ_0 and $\bar{\tau}_0$ respectively, of course).

Consider the alignment $A := (M, \phi) : L_1 \rightsquigarrow L_2$ on Y. The commutativity of (3) precisely means $f^*A \simeq \bar{A}$ (see Remark 1.6). □

4.5. **Corollary.** *Let $f : \bar{Y} \to Y$ be a proper morphism in $\mathcal{S}_X$. Let L_1 and L_2 be line bundles on Y and $\bar{A} : \omega_f \otimes f^*L_1 \rightsquigarrow \omega_f \otimes f^*L_2$ be an alignment on $\bar{Y}$. Suppose that $[L_1] = [L_2]$ in $\mathrm{Pic}_X(Y)/2$. Then there exists an alignment $A : L_1 \rightsquigarrow L_2$ on Y and an isomorphism $f^!A \simeq \bar{A}$ on $\bar{Y}$ (Remark 1.6).*

Proof. Immediate from Theorem 4.4 since $f^! = \omega_f \otimes f^* : \mathcal{A}\ell(Y) \to \mathcal{A}\ell(\bar{Y})$ and since $\bar{B} \mapsto \omega_f \otimes \bar{B}$ is a bijection by Lemma 1.5. □

The following two results turn the above descent of line bundle alignments into descent of alignment isomorphisms on the level of Witt groups.

4.6. **Proposition.** *Let $f : \bar{Y} \to Y$ be a morphism of schemes in $\mathcal{S}_X$. Let L_1 and L_2 be line bundles on Y with $[L_1] = [L_2] \in \mathrm{Pic}_X(Y)/2$. Let $\bar{A}_1 : f^*L_1 \rightsquigarrow \bar{L}$ and $\bar{A}_2 : f^*L_2 \rightsquigarrow \bar{L}$ be two alignments with same target $\bar{L}$ on $\bar{Y}$. Then there exists an alignment $A : L_1 \rightsquigarrow L_2$ on Y such that the following diagram commutes*

$$
\begin{tikzcd}
\mathrm{W}^*_Z(Y, L_1) \arrow[rr, dashed, "A^{\circlearrowleft}"] \arrow[dr, "\mathrm{Pull}_{\bar{A}_1, f}"'] & & \mathrm{W}^*_Z(Y, L_2) \arrow[dl, "\mathrm{Pull}_{\bar{A}_2, f}"] \\
& \mathrm{W}^*_Z(\bar{Y}, \bar{L}) &
\end{tikzcd}
$$

Proof. By Lemma 1.3, there exists an alignment $\bar{A} : f^*L_1 \rightsquigarrow f^*L_2$ on $\bar{Y}$ such that $\bar{A}_2 \circ \bar{A} \simeq \bar{A}_1$. By Theorem 4.4, there exists $A : L_1 \rightsquigarrow L_2$ such that $f^*A \simeq \bar{A}$. So, $\bar{A}_2 \circ f^*A \simeq \bar{A}_1$. By Propositions 3.6 and 2.7 (a), we have $\mathrm{Pull}_{\bar{A}_2, f} \circ A^{\circlearrowleft} = \mathrm{Pull}_{\bar{A}_2 \circ f^*A, f} = \mathrm{Pull}_{\bar{A}_1, f}$. □

4.7. **Proposition.** *Let $f : \bar{Y} \to Y$ be a proper morphism in $\mathcal{S}_X$. Let L_1 and L_2 be line bundles on Y such that $[L_1] = [L_2] \in \mathrm{Pic}_X(Y)/2$. Let $\bar{A}_1 : \bar{L} \rightsquigarrow \omega_f \otimes f^*L_1$ and $\bar{A}_2 : \bar{L} \rightsquigarrow \omega_f \otimes f^*L_2$ be two alignments from the same source $\bar{L}$ on $\bar{Y}$. Then there exists an alignment $A : L_1 \rightsquigarrow L_2$ on Y such that the following diagram commutes*

$$
\begin{tikzcd}
& \mathrm{W}^*_Z(\bar{Y}, \bar{L}) \arrow[dl, "\mathrm{Push}_{f, \bar{A}_1}"'] \arrow[dr, "\mathrm{Push}_{f, \bar{A}_2}"] & \\
\mathrm{W}^*_Z(Y, L_1) \arrow[rr, dashed, "A^{\circlearrowleft}"] & & \mathrm{W}^*_Z(Y, L_2) .
\end{tikzcd}
$$

Proof. By Corollary 4.5 for $\bar{A} = \bar{A}_2 \circ \bar{A}_1^{-1}$, there exists $A : L_1 \rightsquigarrow L_2$ such that $f^!A \circ \bar{A}_1 \simeq \bar{A}_2$. By Proposition 3.6, $A^{\circlearrowleft} \circ \mathrm{Push}_{f, \bar{A}_1} = \mathrm{Push}_{f, f^!A \circ \bar{A}_1} = \mathrm{Push}_{f, \bar{A}_2}$. □

4.8. **Remark.** Combined with Corollary 3.2, the last two propositions tell us that, for a morphism $f : \bar{Y} \to Y$ in $\mathcal{S}_X$, it is not so important to know where a lax pull-back $\mathrm{Pull}_{\bar{A},f}$ or a lax push-forward $\mathrm{Push}_{f,\bar{A}}$ exactly lands, as long as we keep track of classes in $\mathrm{Pic}_X(?)/2$. Different choices can always be "realigned".

5. Relative alignments and lax module structure

Now that we have a stable understanding of alignments, we introduce a relative version of this notion, allowing a line bundle on the base X to intervene.

5.1. **Definition.** Let $\pi_Y : Y \to X$ be a scheme over X. We say that two line bundles L_1 and L_2 over Y are *(quadratically) X-aligned* if $[L_1] = [L_2] \in \mathrm{Pic}_X(Y) = \mathrm{coker}(\pi_Y^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)/2)$. This amounts to the existence of a line bundle K over X and an alignment $\pi_Y^* K \otimes L_1 \rightsquigarrow L_2$ as in Definition 1.1.

By extension, it will be very convenient to say that a Witt class $w \in \mathrm{W}_Z^*(Y, L_1)$ is *X-aligned with* L_2 when the line bundle L_1 is X-aligned with L_2.

Specifying a line bundle K over X, we say that L_1 and L_2 are *K-aligned* if $\pi_Y^* K \otimes L_1$ is aligned with L_2. Unfolding everything, this means that there exists an alignment $A = (M, \phi) : \pi_Y^* K \otimes L_1 \rightsquigarrow L_2$, i.e. an isomorphism $\phi : M^{\otimes 2} \otimes \pi_Y^* K \otimes L_1 \xrightarrow{\sim} L_2$. We call A a *K-alignment of L_1 with L_2* and use the condensed notation

$$A : L_1 \underset{K}{\rightsquigarrow} L_2 \,.$$

5.2. **Definition.** Let $A : L_1 \underset{K}{\rightsquigarrow} L_2$ be a K-alignment in Y. We are going to define a *lax product* or *product realigned under A*

$$- \underset{A}{\cdot} - \ : \ \mathrm{W}^i(X, K) \times \mathrm{W}_Z^j(Y, L_1) \longrightarrow \mathrm{W}_Z^{i+j}(Y, L_2) \,.$$

By Definition 2.3, there exists an alignment isomorphism $A^{\circlearrowleft} : \mathrm{W}_Z^*(Y, \pi_Y^* K \otimes L_1) \xrightarrow{\sim} \mathrm{W}_Z^*(Y, L_2)$. Then for every Witt class $\lambda \in \mathrm{W}^i(X, K)$ on the base and every Witt class $w \in \mathrm{W}_Z^j(Y, L_1)$ on Y, we define

$$\lambda \underset{A}{\cdot} w \ := \ A^{\circlearrowleft}\big(\pi_Y^*(\lambda) \cdot w\big)$$

for the image in $\mathrm{W}_Z^{i+j}(Y, L_2)$ of the product $\pi_Y^*(\lambda) \cdot w$, under the alignment isomorphism $A^{\circlearrowleft}$. We call this the *lax-structure of $\mathrm{W}^{\mathrm{tot}}(X)$-module on $\mathrm{W}_Z^{\mathrm{tot}}(Y)$*.

5.3. **Remark.** Taking $\lambda = 1 \in \mathrm{W}(X)$, we see that $\lambda \cdot_A - = A^{\circlearrowleft}(-)$. So the above homomorphisms $\lambda \cdot_A -$ generalize the alignment isomorphisms.

This action behaves nicely with respect to all possible alignment isomorphisms:

5.4. **Lemma.** *Let $A : L_1 \underset{K}{\rightsquigarrow} L_2$ be a K-alignment so that we have the product $\cdot_A$ of Definition 5.2. Let $B : L_2 \rightsquigarrow L_3$, $C : J \rightsquigarrow K$ and $D : L_0 \rightsquigarrow L_1$ be (plain) alignments over Y, X and Y respectively. Then $E := B \circ A \circ \big((\pi_Y^* C) \otimes D\big)$ is a J-alignment $L_0 \underset{J}{\rightsquigarrow} L_3$ on Y. For every $\lambda \in \mathrm{W}^i(X, J)$ and $w \in \mathrm{W}_Z^j(Y, L_0)$ we have*

$$B^{\circlearrowleft}\Big(C^{\circlearrowleft}(\lambda) \underset{A}{\cdot} D^{\circlearrowleft}(w)\Big) = \lambda \underset{E}{\cdot} w$$

in $\mathrm{W}_Z^{i+j}(Y, L_3)$. In words, the lax product commutes with lax-similitude.

Proof. A direct computation :

$$
\begin{aligned}
B^{\circlearrowleft}\Big(C^{\circlearrowleft}(\lambda)\underset{A}{\cdot}D^{\circlearrowleft}(w)\Big) &\overset{\text{def}}{=} B^{\circlearrowleft}\circ A^{\circlearrowleft}\big(\pi_Y^*(C^{\circlearrowleft}(\lambda))\cdot D^{\circlearrowleft}(w)\big)\\
&\overset{3.2}{=} B^{\circlearrowleft}\circ A^{\circlearrowleft}\Big(\big((\pi_Y^*C)^{\circlearrowleft}(\pi_Y^*(\lambda))\big)\cdot D^{\circlearrowleft}(w)\Big)\\
&\overset{2.7(c)}{=} B^{\circlearrowleft}\circ A^{\circlearrowleft}\circ\big((\pi_Y^*C)\otimes D\big)^{\circlearrowleft}\,(\pi_Y^*(\lambda)\cdot w)\\
&\overset{2.7(b)}{=} \Big(B\circ A\circ\big((\pi_Y^*C)\otimes D\big)\Big)^{\circlearrowleft}\,(\pi_Y^*(\lambda)\cdot w)\overset{\text{def}}{=}\lambda\underset{E}{\cdot}w\,. \qquad\square
\end{aligned}
$$

The real question is whether this product $\lambda\cdot_A w$ depends significantly on the K-alignment $A: L_1 \rightsquigarrow L_2$, for L_1 and L_2 fixed. A priori, this might be the case. However, our class of schemes $\mathcal{S}_X$ (Definition 4.1) turns out to be well-behaved.

5.5. **Lemma.** *Let Y be a scheme in $\mathcal{S}_X$. Let $A_i : L_1 \underset{K_i}{\rightsquigarrow} L_2$ be K_i-alignments over Y (for the same L_1 and L_2), for $i=1,2$. Then there exits an alignment $C: K_1\rightsquigarrow K_2$ on X such that, for every $\lambda_1\in \mathrm{W}^*(X,K_1)$ and every $w\in \mathrm{W}^*_Z(Y,L_1)$, we have*

$$\lambda_1\underset{A_1}{\cdot}w=\lambda_2\underset{A_2}{\cdot}w$$

in $\mathrm{W}^*_Z(Y,L_2)$, where $\lambda_2=C^{\circlearrowleft}(\lambda_1)\in \mathrm{W}^*(X,K_2)$.

Proof. In terms of plain alignments over Y, note that $A_1:\pi_Y^*K_1\otimes L_1\rightsquigarrow L_2$ and $A_2:\pi_Y^*K_2\otimes L_1\rightsquigarrow L_2$ have the same target L_2. By Lemma 1.3, there exists an alignment $A':\pi_Y^*K_1\otimes L_1\rightsquigarrow \pi_Y^*K_2\otimes L_1$ such that $A_2\circ A'\simeq A_1$. Note that L_1 appears at the two ends of A'. So, by Lemma 1.5, there exists an alignment $A'':\pi_Y^*K_1\rightsquigarrow\pi_Y^*K_2$ such that $A'\simeq A''\otimes L_1$. Finally, by Proposition 4.6 applied to $f=\pi_Y$, there exists an alignment $C:K_1\rightsquigarrow K_2$ such that $A''\simeq\pi_Y^*C$. The result follows from Lemma 5.4, applied to our C, and to $\bar B:=\mathrm{id}_{L_2}$, $\bar D:=\mathrm{id}_{L_1}$, $A:=A_2$, checking that E is here $A_2\circ((\pi_Y^*C)\otimes L_1)\simeq A_2\circ(A''\otimes L_1)\simeq A_2\circ A'\simeq A_1$. $\square$

Let us a say a word about associativity of the lax-action.

5.6. **Lemma.** *Let Y be an X-scheme and K_1 and K_2 be line bundles on X. Set $K_3:=K_2\otimes K_1$. Consider X-alignments $A_1: L_0\underset{K_1}{\rightsquigarrow}L_1$ and $A_2: L_1\underset{K_2}{\rightsquigarrow}L_2$ and $A_3: L_0\underset{K_3}{\rightsquigarrow}L_2$ over Y. Then for any choice of two out of A_1, A_2 and A_3, the third can be constructed such that the following diagram commutes in $\mathcal{A}\ell(Y)$:*

$$
\begin{tikzcd}
\pi_Y^*(K_2\otimes K_1)\otimes L_0 \arrow[rr, squiggly, "1\otimes A_1"] \arrow[dr, squiggly, "A_3"'] & & \pi_Y^*K_2\otimes L_1 \arrow[dl, squiggly, "A_2"] \\
& L_2 &
\end{tikzcd}
\tag{4}
$$

Then, for every $w\in \mathrm{W}^j_Z(Y,L_0)$, $\lambda_1\in\mathrm{W}^{i_1}(X,K_1)$ and $\lambda_2\in\mathrm{W}^{i_2}(X,K_2)$, we have

$$\lambda_2\underset{A_2}{\cdot}\big(\lambda_1\underset{A_1}{\cdot}w\big)=\big(\lambda_2\cdot\lambda_1\big)\underset{A_3}{\cdot}w$$

in $\mathrm{W}^{i_1+i_2+j}_Z(Y,L_2)$.

Proof. The first part follows from Lemmas 1.3 and 1.5. The rest is direct :

$$
\begin{aligned}
\lambda_2\cdot_{A_2}\big(\lambda_1\cdot_{A_1}w\big) &= A_2{}^{\circlearrowleft}\big(\pi_Y^*\lambda_2\cdot A_1{}^{\circlearrowleft}(\pi_Y^*\lambda_1\cdot w)\big) && \text{by definition}\\
&= A_2{}^{\circlearrowleft}\circ(1\otimes A_1)^{\circlearrowleft}\big(\pi_Y^*\lambda_2\cdot\pi_Y^*\lambda_1\cdot w\big) && \text{by Proposition 2.7 (c)}\\
&= A_3{}^{\circlearrowleft}(\pi_Y^*\lambda_2\cdot\pi_Y^*\lambda_1\cdot w) && \text{by (4) and Prop. 2.7}\\
&= A_3{}^{\circlearrowleft}\big(\pi_Y^*(\lambda_2\cdot\lambda_1)\cdot w\big) && \pi^*\text{ is a ring morphism}\\
&= \big(\lambda_2\cdot\lambda_1\big)\cdot_{A_3}w && \text{by definition.}\quad\square
\end{aligned}
$$

Let us now discuss the lax-linearity of lax pull-back and lax push-forward.

5.7. **Lemma.** *Let $f:\bar{Y}\to Y$ be a morphism in $\mathcal{S}_X$ and $Z\subset Y$ be closed. Consider two lax pull-backs (Definition 3.4):*

$$\mathrm{Pull}_{\bar{A},f}:\mathrm{W}^*_Z(Y,L)\to\mathrm{W}^*_{f^{-1}Z}(\bar{Y},\bar{L})\quad\text{and}\quad\mathrm{Pull}_{\bar{B},f}:\mathrm{W}^*_Z(Y,M)\to\mathrm{W}^*_{f^{-1}Z}(\bar{Y},\bar{M})$$

for two alignments $\bar{A}:f^*L\rightsquigarrow\bar{L}$ and $\bar{B}:f^*M\rightsquigarrow\bar{M}$ over $\bar{Y}$. Suppose that the line bundles L and M over Y are X-aligned, i.e. $[L]=[M]$ in $\mathrm{Pic}_X(Y)/2$.

(a) For every K-alignment $C:L\underset{K}{\rightsquigarrow}M$ over Y, there exists a K-alignment $\bar{C}:\bar{L}\underset{K}{\rightsquigarrow}\bar{M}$ over $\bar{Y}$ such that for all $\lambda\in\mathrm{W}^*(X,K)$ and all $w\in\mathrm{W}^*_Z(Y,L)$

$$\mathrm{Pull}_{\bar{B},f}\left(\lambda\underset{C}{\cdot}w\right)=\lambda\underset{\bar{C}}{\cdot}\left(\mathrm{Pull}_{\bar{A},f}(w)\right)\qquad\text{in }\mathrm{W}^*_{f^{-1}Z}(\bar{Y},\bar{M})\,.$$

(b) For every K-alignment $\bar{C}:\bar{L}\underset{K}{\rightsquigarrow}\bar{M}$ over $\bar{Y}$, there exists a K-alignment $C:L\underset{K}{\rightsquigarrow}M$ over Y such that the very same equation holds (maybe better read from right to left this time).

Proof. For (a), use Lemma 1.3 to construct $\bar{C}$ such that $\bar{B}\circ f^*C\simeq\bar{C}\circ(\mathrm{id}\otimes\bar{A})$, i.e. solve the following left-hand square:

$$\begin{tikzcd}
\pi_{\bar{Y}}^*K\otimes f^*L \arrow[r, squiggly, "f^*C"] \arrow[d, squiggly, "\mathrm{id}\otimes\bar{A}"'] & f^*M \arrow[d, squiggly, "\bar{B}"] \\
\pi_{\bar{Y}}^*K\otimes\bar{L} \arrow[r, squiggly, "\exists\,\bar{C}"] & \bar{M}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\pi_{\bar{Y}}^*K\otimes f^*L \arrow[r, squiggly, "\exists\,\bar{D}"] \arrow[d, squiggly, "\mathrm{id}\otimes\bar{A}"'] & f^*M \arrow[d, squiggly, "\bar{B}"] \\
\pi_{\bar{Y}}^*K\otimes\bar{L} \arrow[r, squiggly, "\bar{C}"] & \bar{M}
\end{tikzcd}$$

For (b), first solve the above right-hand square to find $\bar{D}$ and use Theorem 4.4 to find $C:\pi_Y^*K\otimes L\rightsquigarrow M$ such that $\bar{D}\simeq f^*C$. In both cases we have

$$\bar{B}\circ f^*C\simeq\bar{C}\circ(\mathrm{id}\otimes\bar{A})\,. \tag{5}$$

Then compute

$$\begin{aligned}
\mathrm{Pull}_{\bar{B},f}\left(\lambda\cdot_C w\right) &= \bar{B}^{\circlearrowleft}\circ f^*\circ C^{\circlearrowleft}(\pi_Y^*(\lambda)\cdot w) && \text{by definition}\\
&= \bar{B}^{\circlearrowleft}\circ(f^*C)^{\circlearrowleft}\circ f^*(\pi_Y^*(\lambda)\cdot w) && \text{by Corollary 3.2}\\
&= \bar{B}^{\circlearrowleft}\circ(f^*C)^{\circlearrowleft}\left(\pi_{\bar{Y}}^*(\lambda)\cdot f^*(w)\right) && f^*\text{ is a ring homomorphism}\\
&= \bar{C}^{\circlearrowleft}\circ(\mathrm{id}\otimes\bar{A})^{\circlearrowleft}\left(\pi_{\bar{Y}}^*(\lambda)\cdot f^*(w)\right) && \text{by (5) and Proposition 2.7}\\
&= \bar{C}^{\circlearrowleft}\left(\pi_{\bar{Y}}^*(\lambda)\cdot(\bar{A}^{\circlearrowleft}\circ f^*(w))\right) && \text{by Proposition 2.7 (c)}\\
&= \lambda\cdot_{\bar{C}}\left(\mathrm{Pull}_{\bar{A},f}(w)\right)\,. && \text{by definition.}
\end{aligned}$$
□

5.8. **Lemma.** *Let $f:\bar{Y}\to Y$ be a proper morphism in $\mathcal{S}_X$, of constant relative dimension, and $\bar{Z}\subset\bar{Y}$ be closed. Consider two lax push-forwards (Definition 3.5):*

$$\mathrm{Push}_{f,\bar{A}}:\mathrm{W}^\star_{\bar{Z}}(\bar{Y},\bar{L})\to\mathrm{W}^*_{f\bar{Z}}(Y,L)\quad\text{and}\quad\mathrm{Push}_{f,\bar{B}}:\mathrm{W}^\star_{\bar{Z}}(\bar{Y},\bar{M})\to\mathrm{W}^*_{f\bar{Z}}(Y,M)$$

where $\star=*+\dim f$ for two alignments $\bar{A}:\bar{L}\rightsquigarrow\omega_f\otimes f^*L$ and $\bar{B}:\bar{M}\rightsquigarrow\omega_f\otimes f^*M$ over $\bar{Y}$. Suppose that the line bundles L and M over Y are X-aligned, i.e. $[L]=[M]$ in $\mathrm{Pic}_X(Y)/2$.

(a) For every K-alignment $C:L\underset{K}{\rightsquigarrow}M$ over Y, there exists a K-alignment $\bar{C}:\bar{L}\underset{K}{\rightsquigarrow}\bar{M}$ over $\bar{Y}$ such that for all $\bar{w}\in\mathrm{W}^\star_{\bar{Z}}(\bar{Y},\bar{L})$ and $\lambda\in\mathrm{W}^*(X,K)$ we have

$$\lambda\underset{C}{\cdot}\left(\mathrm{Push}_{f,\bar{A}}(\bar{w})\right)=\mathrm{Push}_{f,\bar{B}}\left(\lambda\underset{\bar{C}}{\cdot}\bar{w}\right)\qquad\text{in }\mathrm{W}^*_{f\bar{Z}}(Y,M)\,.$$

(b) For every K-alignment $\bar{C}:\bar{L}\underset{K}{\rightsquigarrow}\bar{M}$ over $\bar{Y}$, there exists a K-alignment $C:L\underset{K}{\rightsquigarrow}M$ over Y such that the very same property holds (read backwards).

Proof. For (a), use Lemma 1.3 to construct $\bar{C}$ such that the following left-hand square commutes:

$$\begin{tikzcd}
\pi_Y^* K \otimes \bar{L} \arrow[r, rightsquigarrow, dashed, "\exists \bar{C}"] \arrow[d, rightsquigarrow, "\bar{A}' := (12)\circ(\mathrm{id}\otimes\bar{A})"'] & \bar{M} \arrow[d, rightsquigarrow, "\bar{B}"] \\
\omega_f \otimes \pi_Y^* K \otimes f^* L \arrow[r, rightsquigarrow, "f^! C"] & \omega_f \otimes f^* M
\end{tikzcd}
\qquad
\begin{tikzcd}
\pi_Y^* K \otimes \bar{L} \arrow[r, rightsquigarrow, "\bar{C}"] \arrow[d, rightsquigarrow, "\bar{A}'"'] & \bar{M} \arrow[d, rightsquigarrow, "\bar{B}"] \\
\omega_f \otimes \pi_Y^* K \otimes f^* L \arrow[r, rightsquigarrow, dashed, "\exists \bar{D}"] & \omega_f \otimes f^* M
\end{tikzcd}$$

For (b), first solve the above right-hand square to find $\bar{D}$ and use Corollary 4.5 to find $C : \pi_Y^* K \otimes L \rightsquigarrow M$ such that $\bar{D} \simeq f^! \bar{C}$. In both cases we have

$$\bar{B} \circ \bar{C} \simeq f^! C \circ \bar{A}' . \tag{6}$$

Then compute

$$\begin{aligned}
\mathrm{Push}_{f,\bar{B}}\left(\lambda \cdot_{\bar{C}} \bar{w}\right) &= f_* \circ \bar{B}^{\circlearrowleft} \circ \bar{C}^{\circlearrowleft}(\pi_Y^*(\lambda) \cdot \bar{w}) && \text{by definition} \\
&= f_* \circ (f^! C)^{\circlearrowleft} \circ \bar{A}'^{\circlearrowleft}(\pi_Y^*(\lambda) \cdot \bar{w}) && \text{by (6) and Proposition 2.7} \\
&= C^{\circlearrowleft} \circ f_* \circ \bar{A}'^{\circlearrowleft}(\pi_Y^*(\lambda) \cdot \bar{w}) && \text{by Corollary 3.2} \\
&= C^{\circlearrowleft} \circ f_*\Big(\underline{(12)}\big(\pi_Y^*(\lambda) \cdot \bar{A}^{\circlearrowleft}(\bar{w})\big)\Big) && \text{by Proposition 2.7 (c)} \\
&= C^{\circlearrowleft}\Big(\pi_Y^*(\lambda) \cdot f_*\big(\bar{A}^{\circlearrowleft}(\bar{w})\big)\Big) && \text{by projection formula for } f_* \\
&= \lambda \cdot_C \left(\mathrm{Push}_{f,\bar{A}}(\bar{w})\right). && \text{by definition.}
\end{aligned}$$

The permutation of line bundles $\underline{(12)}$ in the fourth equation is usually dropped but actually is the precise way to state the projection formula. $\square$

5.9. **Remark.** Following up on Remark 2.10, it is easy to verify that the lax module structure is compatible with the localization long exact sequence.

6. Total bases of the total Witt group

We want to define what should be a basis of the non-existent total Witt group of Y with support in Z, over the similarly evanescent total Witt group of X. The intuitive meaning is simple. We want every Witt class of $\mathrm{W}^*(Y, L)$ to be a sum of lax-products of elements of the basis by Witt classes over X (the coefficients) and we want no linear relation among the Witt classes in the basis, with coefficients over X.

6.1. **Setup.** We will repeatedly use the following situation: Let $L_1, \ldots, L_n$ be line bundles over an X-scheme Y, let $j_1, \ldots, j_n$ be integers and $Z \subset Y$ a closed subset. We consider Witt classes $w_1, \ldots, w_n$ where each $w_i \in \mathrm{W}_Z^{j_i}(Y, L_i)$ lives in its own Witt group of Y with support in the common closed subset $Z \subset Y$. We want to make sense of linear combinations of $w_1, \ldots, w_n$ with coefficients in the Witt groups of X. With the lax module structure of Section 5, there are many ways to multiply each w_i by a coefficient λ_i over X. We clarify this first.

6.2. **Definition.** Let $w_1, \ldots, w_n$ be Witt classes over Y as in 6.1 and assume they are X*-aligned* in the sense of Definition 5.1, i.e. $[L_1] = \cdots = [L_n]$ in $\mathrm{Pic}_X(Y)/2$.

A set of *compatible coefficients* for $w_1, \ldots, w_n$ consists of two ingredients:

- Witt classes $\lambda_1 \in \mathrm{W}^{i_1}(X, K_1), \ldots, \lambda_n \in \mathrm{W}^{i_n}(X, K_n)$ over X, with the property that $i_1 + j_1 = i_2 + j_2 = \cdots = i_n + j_n$; call this number $k \in \mathbb{Z}$.
- A K_i-alignment $C_i : L_i \underset{K_i}{\rightsquigarrow} L$ (Definition 5.1), for every $i = 1, \ldots, n$, for a common line bundle L over Y; that is, a pair $C_i = (M_i, \phi_i)$ where M_i is a line bundle on Y and $\phi_i : M_i^{\otimes 2} \otimes \pi_Y^* K_i \otimes L_i \xrightarrow{\sim} L$ is an isomorphism.

When $k \in \mathbb{Z}$ and the line bundle L over Y are specified in advance, we speak of (k, L)-*compatible coefficients*. Naturally, we abbreviate all this by writing that the "$\lambda_1, \ldots, \lambda_n$ are compatible coefficients for $w_1, \ldots, w_n$". We also use the mildly abusive notation $\lambda_i w_i$ for $\lambda_i \cdot_{C_i} w_i$ when there is no risk of confusion, but we insist that the alignments C_i come with the coefficients λ_i's in any case, possibly implicitly. This lax product $\lambda_i \cdot_{C_i} w_i$ belongs to $\mathrm{W}^k(Y, L)$. We then define the *lax linear combination* of the $w_1, \ldots, w_n$ with coefficients $\lambda_1, \ldots, \lambda_n$ as the following element in $\mathrm{W}^k_Z(Y, L)$:

$$\sum \lambda_i w_i := \sum \lambda_i \underset{C_i}{\cdot} w_i \,.$$

6.3. **Definition.** Let $Z \subset Y$ be closed and let $\mathcal{I}$ be a set. A family $(w_i)_{i \in \mathcal{I}}$ of Witt classes $w_i \in \mathrm{W}^{j(i)}_Z(Y, L_i)$ is called *totally independent over* X if for every finite subset $\mathcal{J}$ of $\mathcal{I}$ such that the $(w_i)_{i \in \mathcal{J}}$ are X-aligned and every compatible coefficients $(\lambda_i)_{i \in \mathcal{J}}$, the relation $\sum_{\mathcal{J}} \lambda_i w_i = 0$ forces all λ_i to be zero.

6.4. **Remark.** In the following definitions, we are going to use a subset P of $\mathrm{Pic}_X(Y)/2$. The reader might want to assume at first that P is the whole $\mathrm{Pic}_X(Y)/2$ for this will often be the case. Allowing other P's will only become relevant when dealing with the functorial behavior of these notions, and only in "fringe cases". If $[L] \in P$, we say that L is X-*aligned with* P and we also say that every Witt class $w \in \mathrm{W}^*_Z(Y, L)$ is X-*aligned with* P. These conditions are empty for $P = \mathrm{Pic}_X(Y)$.

6.5. **Definition.** Let $Z \subset Y$ be closed and P be a subset of $\mathrm{Pic}_X(Y)/2$. Let $(w_i)_{i \in \mathcal{I}}$ be a family of Witt classes over Y with support in Z, which are all X-aligned with P. We say that $(w_i)_{i \in \mathcal{I}}$ *totally generates the P-part of the Witt groups of Y with support in Z, over X*, if for every line bundle L over Y such that $[L] \in P$, every integer k and every $y \in \mathrm{W}^k_Z(Y, L)$, there exists a finite subset $\mathcal{J}$ of $\mathcal{I}$ such that $(w_i)_{i \in \mathcal{J}}$ are aligned with L, and (k, L)-compatible coefficients $(\lambda_i)_{i \in \mathcal{J}}$ over X such that $y = \sum_{i \in \mathcal{J}} \lambda_i w_i$ as in Definition 6.2.

6.6. **Definition.** Let $P \subset \mathrm{Pic}_X(Y)/2$ and $Z \subset Y$ closed. We say that a family $(w_i)_{i \in \mathcal{I}}$ of Witt classes X-aligned with P forms a *total basis* of the P-part of the Witt groups of Y with support in Z, over X, if it is totally independent (Definition 6.3) and totally generates (Definition 6.5).

6.7. **Example.** For $Z = Y = X$, the unit $1 \in \mathrm{W}^0(X, \mathcal{O}_X)$ is a total basis over X.

6.8. **Remark.** Unlike the classical notion, total independence does not strictly imply uniqueness of coefficients in a linear combination; given totally independent classes $w_1, \ldots, w_n$, all X-aligned with some line bundle L, we could have $\sum \lambda_i \cdot_{C_i} w_i = \sum \lambda'_i \cdot_{C'_i} w_i$ without $\lambda_i = \lambda'_i$ for all i. Equality only follows from independence if the alignments C_i and C'_i are the same. However, Lemma 5.5 tells us that if Y is in $\mathcal{S}_X$, we can find an alignment $A_i : K_i \rightsquigarrow K'_i$ over X for every i such that $\lambda_i \cdot_{C_i} w_i = \lambda''_i \cdot_{C_i} w_i$ with $\lambda''_i = A_i{}^{\circlearrowleft}(\lambda'_i)$. Then, we must have $\lambda_i = \lambda''_i$ by total independence.

Anyway, for X-schemes in our class $\mathcal{S}_X$ (Definition 4.1), we have the following "classical" interpretation of a total basis:

6.9. **Proposition.** *Let $Y \in \mathcal{S}_X$. Let $P \subset \mathrm{Pic}_X(Y)/2$ be a subset, $Z \subset Y$ closed and let $\left(w_i \in \mathrm{W}^{j_i}_Z(Y, L_i)\right)_{i \in \mathcal{I}}$ be a set of Witt classes on Y with support in Z, such that each $[L_i] \in P$. For each $p \in P$ set $\mathcal{I}_p = \left\{ i \in \mathcal{I} \,\middle|\, [L_i] = p \text{ in } \mathrm{Pic}_X(Y)/2 \right\}$. Then the following properties are equivalent:*

(i) The family $(w_i)_{i \in \mathcal{I}}$ is a total basis of the P-part of the Witt groups of Y with support in Z, over X (Definition 6.6).

(ii) *For every line bundle L with $[L] \in P$, every $k \in \mathbb{Z}$ and* for every choice, *for those $i \in \mathcal{I}_{[L]}$, of a line bundle K_i over X and a K_i-alignment $C_i : L_i \underset{K_i}{\rightsquigarrow} L$, the following map is an isomorphism*

$$\begin{aligned} \theta = \theta((C_i)_i) \ : \bigoplus_{i \in \mathcal{I}_{[L]}} \mathrm{W}^{k-j_i}(X, K_i) \quad &\xrightarrow{\sim} \quad \mathrm{W}^k_Z(Y, L) \\ (x_i)_{i \in \mathcal{I}_{[L]}} \quad &\longmapsto \quad \sum x_i \cdot_{C_i} w_i \,. \end{aligned} \tag{7}$$

(iii) *For every class $p \in P$ and every $k \in \mathbb{Z}$, there exists a choice of $L \in p$ and* there exists a choice, *for each $i \in \mathcal{I}_p$, of a line bundle K_i over X and a K_i-alignment $C_i : L_i \underset{K_i}{\rightsquigarrow} L$ for which (7) is an isomorphism.*

Note that θ as in (7) is always a homomorphism of $\mathrm{W}(X)$*-modules by Lemma 5.6.*

Proof. $(i) \Rightarrow (ii)$: Injectivity is straightforward from total independence (Definition 6.3). For surjectivity, let $y \in \mathrm{W}^k_Z(Y, L)$ and use total generation (Definition 6.5) to write y as $\sum_{i \in \mathcal{J}} \lambda_i \cdot_{C_i} w_i$ for some finite subset $\mathcal{J} \subset \mathcal{I}$, some coefficients $\lambda_i \in \mathrm{W}^{k-j_i}(X, J_i)$ and some J_i-alignments $D_i : L_i \underset{J_i}{\rightsquigarrow} L$, for $i \in \mathcal{J}$. A priori, J_i might differ from K_i and D_i might differ from C_i. But Lemma 5.5 tells us that each $\lambda_i \cdot_{D_i} w_i = x_i \cdot_{C_i} w_i$ for a suitable $x_i \in \mathrm{W}^*(X, K_i)$ lax-similar to λ_i . Hence $y = \sum_{i \in \mathcal{J}} x_i \cdot_{C_i} w_i \in \mathrm{im}(\theta)$. So, θ is surjective.

$(ii) \Rightarrow (iii)$: Do choose $L \in p$ and X-alignments $C_i : L_i \underset{K_i}{\rightsquigarrow} L$ for $i \in \mathcal{I}_p$.

$(iii) \Rightarrow (i)$: Total generation is immediate from surjectivity of θ and Lemma 5.5. For total independence, let $(w_i)_{i \in \mathcal{J}}$ be X-aligned (as in 6.1) and let $(\lambda_i)_{i \in \mathcal{J}}$ be (L', k)-compatible coefficients for some L', such that $\sum_{i \in \mathcal{J}} \lambda_i \cdot_{D_i} w_i = 0$ for suitable alignments D_i. Note that $[L'] \in P$. Choose L, K_i and C_i as in (iii) for $p = [L']$. Choose also $A : L' \rightsquigarrow L$. Then $A^{\circlearrowleft}\big(\sum_{i \in \mathcal{J}} \lambda_i \cdot_{D_i} w_i\big) = 0$ as well. By Lemmas 5.4 and 5.5 again, each $A^{\circlearrowleft}(\lambda_i \cdot_{D_i} w_i) = x_i \cdot_{C_i} w_i$ for some x_i lax-similar to λ_i. We then get $\theta((x_i)_{i \in \mathcal{J}}) = 0$ which forces all $x_i = 0$ by injectivity of θ. But then $\lambda_i = 0$ as well since alignment isomorphisms are... isomorphisms. □

6.10. **Lemma.** *Let P and P' be subsets of* $\mathrm{Pic}_X(Y)/2$ *and let $(w_i)_{i \in \mathcal{I}}$ (resp. $(w_i)_{i \in \mathcal{I}'}$) be a totally generating family of the P-part (resp. the P'-part) of the Witt groups of Y with support in Z, over X. Then the union family $(w_i)_{i \in \mathcal{I} \cup \mathcal{I}'}$ is a totally generating family of the $P \cup P'$-part of the Witt groups of Y over X. If P and P' are disjoint and if the families $(w_i)_{i \in \mathcal{I}}$ and $(w_i)_{i \in \mathcal{I}'}$ are both totally independent, then their union is totally independent.*

Proof. Clear. □

6.11. **Remark.** Given $i \equiv j$ modulo 4, there is a canonical isomorphism $\mathrm{W}^i \xrightarrow{\sim} \mathrm{W}^j$, given by $(\Sigma^2)^{\frac{j-i}{4}}$, which involves no choice and no sign. Moreover, this isomorphism commutes with every pull-back, push-forward, alignment isomorphism and products (still no sign because $\frac{j-i}{2}$ is even). In other words, a Witt class $w \in \mathrm{W}^i$ corresponds to a unique Witt class of W^j.

If one has an X-scheme Y and a family of Witt classes on Y, one can wonder whether the notions of total independence and total generation (Definitions 6.3 and 6.5) would be different if one identified every Witt class $w \in \mathrm{W}^i$ with its image in W^j, for $j \equiv i$ modulo 4. The answer is no, as long as one does the same on X.

Indeed, $\Sigma^2(\lambda \cdot_A w) = \big(\Sigma^2(\lambda)\big) \cdot_A w$. Hence every occurrence of Σ^2 on Y can be "absorbed" in the coefficients.

The following analogy might help the puzzled reader. If $R = \oplus_{i \in \mathbb{Z}} R^i$ is a $\mathbb{Z}$-graded ring and $M = \oplus_{i \in \mathbb{Z}} M^i$ is a graded R-module (or R-algebra), and if there exists $s \in R^4$ invertible and central (nothing special about 4, of course), then one can consider the $\mathbb{Z}/4$-graded ring $\bar{R} = \oplus_{[i] \in \mathbb{Z}/4} R^i$ with $0 \le i \le 3$ and the graded

$\bar{R}$-module $\bar{M} = \oplus_{[i]\in\mathbb{Z}/4} M^i$ with $0 \le i \le 3$, where a product taking values outside of the range $0 \le i \le 3$ is brought back in that range by using the *unique* power of s which does the job. The point is that a collection $\mathcal{M} \subset \cup_{i\in\mathbb{Z}} M^i$ of homogeneous elements in M form an R-basis of M if and only if the *very same* collection forms an $\bar{R}$-basis of $\bar{M}$ (once brought back in the range $0 \le i \le 3$). In particular $\bar{M}$ has the same dimension over $\bar{R}$ as M had over R. The only possible confusion would come from the perverse contemplation of $\bar{M}$ as an R-module.

Let us now examine how these notions behave under pull-backs or push-forwards. The assumption about injectivity of $f^*_{|P}$ below is the very reason we allow the flexibility of those subsets $P \subset \mathrm{Pic}_X(Y)$, see Remark 6.17.

6.12. **Proposition.** *Let $f : \bar{Y} \to Y$ be a morphism of schemes in $\mathcal{S}_X$. Let P be a subset of $\mathrm{Pic}_X(Y)/2$ and $Z \subset Y$ closed. Suppose that the pull-back map $f^*|_P : P \to \mathrm{Pic}_X(\bar{Y})/2$ is injective, as a map of sets. Suppose also that the pull-back $f^* : \mathrm{W}^k_Z(Y, L) \to \mathrm{W}^k_{f^{-1}(Z)}(\bar{Y}, f^*L)$ is an isomorphism for all L with $[L] \in P$ and all $k \in \mathbb{Z}$.*

Let $(w_i)_{i\in\mathcal{I}}$ be a set of Witt classes $w_i \in \mathrm{W}^{j_i}_Z(Y, L_i)$ over Y with support in Z, with all $[L_i] \in P$. Choose for every i an alignment $\bar{A}_i : f^*L_i \rightsquigarrow \bar{L}_i$ over $\bar{Y}$, hence a lax pull-back $\mathrm{Pull}_{\bar{A}_i,f} : \mathrm{W}^{j_i}_Z(Y, L_i) \to \mathrm{W}^{j_i}_{f^{-1}(Z)}(\bar{Y}, \bar{L}_i)$. Let $\bar{w}_i := \mathrm{Pull}_{\bar{A}_i,f}(w_i)$.

Then $(w_i)_{i\in\mathcal{I}}$ is a total basis of the P-part of the Witt group of Y with support in Z, over X, if and only if $(\bar{w}_i)_{i\in\mathcal{I}}$ is a total basis of the f^*P-part of the Witt group of $\bar{Y}$ with support in $f^{-1}Z$, over X.

Proof. We simply use Proposition 6.9, and its notation, both for Y and for $\bar{Y}$, in the following commutative diagram (for line bundles and alignments to be specified) :

$$\begin{tikzcd}
\bigoplus\limits_{i\in\mathcal{I}_{[L]}} \mathrm{W}^{k-j_i}(X, K_i) \arrow[r, "\theta"] \arrow[d, equal] & \mathrm{W}^k_Z(Y, L) \arrow[d, "\mathrm{Pull}_{\bar{B},f}", "\simeq"'] \\
\bigoplus\limits_{i\in\mathcal{I}_{[\bar{L}]}} \mathrm{W}^{k-j_i}(X, K_i) \arrow[r, "\bar{\theta}"] & \mathrm{W}^k_{f^{-1}Z}(\bar{Y}, \bar{L})
\end{tikzcd} \tag{8}$$

Let $k \in \mathbb{Z}$. Given a line bundle L on Y, we can set $\bar{L} = f^*L$ and $\bar{B} = \mathrm{id}$. Conversely, given $\bar{L}$ on $\bar{Y}$ with $[\bar{L}] \in f^*(P) \subset \mathrm{Pic}_X(Y)/2$, Lemma 4.3 (d) provides an L over Y with $[L] \in P$ such that $f^*[L] = [\bar{L}]$ in $\mathrm{Pic}(\bar{Y})/2$ already. The latter allows us to choose $\bar{B} : f^*L \rightsquigarrow \bar{L}$ a (plain) alignment over $\bar{Y}$, hence to use the lax pull-back $\mathrm{Pull}_{\bar{B},f}$ as on the right-hand side of (8).

Of course, every $\bar{w}_i$ is X-aligned with f^*P. Furthermore, our assumption about the Picard-group f^* being injective on P implies that w_i is X-aligned with L if and only if $\bar{w}_i$ is X-aligned with $\bar{L}$ (use part (c) of 4.3), thus $f^* : \mathcal{I}_{[L]} \overset{\sim}{\to} \mathcal{I}_{[\bar{L}]}$ is a bijection and the left hand side of (8) also makes sense.

Now choose for every $i \in \mathcal{I}_{[L]}$ a K_i-alignment $C_i : L_i \underset{K_i}{\rightsquigarrow} L$, so that we can create $\theta : (x_i) \mapsto \sum_{i\in\mathcal{I}} x_i \cdot_{C_i} w_i$ in (8) as we did in (7). By Lemma 5.7 (a), there exists K_i-alignments $\bar{C}_i : f^*L_i \underset{K_i}{\rightsquigarrow} \bar{L}$ over $\bar{Y}$ such that

$$\mathrm{Pull}_{\bar{B},f}(x \underset{C_i}{\cdot} w_i) = x \underset{\bar{C}_i}{\cdot} \mathrm{Pull}_{\bar{A}_i,f}(w_i)$$

for all $x \in \mathrm{W}^*(X, K_i)$ and all $i \in \mathcal{I}_{[L]}$. So we can define $\bar{\theta}$ by $(x_i) \mapsto \sum x_i \cdot_{\bar{C}_i} \bar{w}_i$, to make (8) commutative. Consequently, θ and $\bar{\theta}$ are simultaneously isomorphisms. By Proposition 6.9, $(w_i)_{i\in\mathcal{I}}$ and $(\bar{w}_i)_{i\in\mathcal{I}}$ are simultaneously bases. $\square$

6.13. **Corollary.** *Hypotheses of Proposition 6.12 hold when $f : \bar{Y} \to Y$ is an affine bundle. So, in that case, a family is a total basis over X of the P-part of the Witt*

groups of Y with support in Z, if and only if, it is pulled-back to a total basis over X of the $f^*(P)$-part of the Witt groups of $\bar{Y}$ with support in $f^{-1}Z$. □

6.14. **Corollary.** *For $Y \in \mathcal{S}_X$, the notions of total independence, total generation, and total basis are stable under alignment isomorphisms (Definition 2.3). For instance, if $(w_i)_{i\in\mathcal{I}}$ is a total basis of the P-part of the Witt group of Y with support in Z, over X, the family $(A_i{}^{\circlearrowleft}(w_i))_{i\in\mathcal{I}}$ is still such a basis for any family of alignment isomorphisms $(A_i{}^{\circlearrowleft})_{i\in\mathcal{I}}$ (e.g. multiplications by a unit of Y, see Example 2.4).*

Proof. Apply Proposition 6.12 to $f = \mathrm{id}_Y$. □

6.15. **Proposition.** *Let f be a proper morphism of schemes in $\mathcal{S}_X$ with constant relative dimension d. Let P be a subset of $\mathrm{Pic}_X(Y)/2$ and $\bar{Z} \subset \bar{Y}$. Let $f^!P := [\omega_f]\cdot f^*P \subseteq \mathrm{Pic}_X(\bar{Y})/2$. Suppose the function $f^*_{|P} : P \to \mathrm{Pic}_X(\bar{Y})/2$ injective.*

Suppose also that for any line bundle L such that $[L] \in P$, the push-forward map $f_* : \mathrm{W}^{k+d}_{\bar{Z}}(\bar{Y}, \omega_f \otimes f^*L) \to \mathrm{W}^k_{f\bar{Z}}(Y, L)$ is an isomorphism for all $k \in \mathbb{Z}$.

Then a family $(\bar{w}_i)_{i\in\mathcal{I}}$ of Witt classes on $\bar{Y}$ with support in $\bar{Z}$, X-aligned with $f^!P$, is a total basis of the $f^!P$-part of the Witt group of $\bar{Y}$ with support in $\bar{Z}$ if and only if the image family $\big(\mathrm{Push}_{f,\bar{A}_i}(\bar{w}_i)\big)_{i\in\mathcal{I}}$ under any family of lax push-forwards corresponding to alignments $(\bar{A}_i)_{i\in\mathcal{I}}$ is a total basis of the P-part of the Witt group of Y with support in $f(Z)$.

Proof. The proof is similar to that of Proposition 6.12, mutatis mutandis. One compares two θ homomorphisms that are “arranged” via Lemma 5.8 this time. □

In particular, the dévissage isomorphism for Witt groups yields the following.

6.16. **Corollary.** *Let $\iota : Z \hookrightarrow Y$ be a closed immersion of constant codimension, with Z and Y in $\mathcal{S}_X$. Let P be a subset of $\mathrm{Pic}_X(Y)/2$ such that the map of sets $\iota^*_{|P} : P \to \mathrm{Pic}_X(Z)/2$ is injective. Let $\iota^!P = [\omega_\iota]\cdot\iota^*P \subseteq \mathrm{Pic}_X(Z)/2$. Let $(w_i)_{i\in\mathcal{I}}$ be elements of the P-part of the total Witt groups of Y with support in Z, and for each $i \in \mathcal{I}$, let v_i be a Witt class in the $\iota^!P$-part of the Witt groups of Z over X such that $w_i = \iota_*(v_i)$ (this is always possible by dévissage). The family $(v_i)_{i\in\mathcal{I}}$ is a total basis of the $\iota^!P$-part of the Witt groups of Z, over X, if and only if the family $(w_i)_{i\in\mathcal{I}}$ is a total basis of the P-part of the Witt groups of Y with support in Z, over X.*

6.17. **Remark.** The injectivity condition on $f^* : P \to \mathrm{Pic}_X(\bar{Y})/2$ is not really harmful because one can always split $\mathrm{Pic}_X(Y)/2$ in smaller P-chunks to ensure that the condition holds for each of them, and then use Lemma 6.10 to obtain a total basis for the whole $P = \mathrm{Pic}_X(Y)/2$. This happens in “fringe” cases, in the cellular decomposition of the Grassmannians, for instance, see [3].

7. Total bases in the localization long exact sequence

Let U be the open complement of a closed subset $Z \subset Y$, and let $\upsilon : U \hookrightarrow Y$ be the corresponding open embedding. Assume both Y and U are in $\mathcal{S}_X$. Let $\mathrm{e} : \mathrm{W}^*_Z(Y, L) \to \mathrm{W}^*(Y, L)$ be the extension of support map. Recall that in this situation, there is a long exact sequence of localization

$$\cdots \longrightarrow \mathrm{W}^i_Z(Y, L) \xrightarrow{\mathrm{e}} \mathrm{W}^i(Y, L) \xrightarrow{\upsilon^*} \mathrm{W}^i(U, \upsilon^*L) \xrightarrow{\partial} \mathrm{W}^{i+1}_Z(Y, L) \to \cdots$$

7.1. **Theorem.** *Let P be a subset of $\mathrm{Pic}_X(Y)/2$. Assume that the restriction $\upsilon^*_{|P} : P \to \mathrm{Pic}_X(U)/2$ is injective and let $P_U = \upsilon^*(P) \subset \mathrm{Pic}_X(U)/2$.*

Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be sets and let $(w'_i)_{i\in\mathcal{I}}$ and $(w_j)_{j\in\mathcal{J}}$ be elements in Witt groups of Y, let $(v_i)_{i\in\mathcal{I}}$ and $(v'_k)_{k\in\mathcal{K}}$ be elements in Witt groups of Y with support in Z

and let $(u_k)_{k\in\mathcal{K}}$ and $(u'_j)_{j\in\mathcal{J}}$ be elements in Witt groups of U, whose line bundles are restricted from Y. (Recall lax-similitude $\leftrightsquigarrow$ from Definition 2.5.) Suppose the following conditions hold (see Figure 1):

(a) for every $i \in \mathcal{I}$, we have $\mathrm{e}(v_i) \leftrightsquigarrow w'_i$
(b) for every $j \in \mathcal{J}$, we have $\upsilon^*(w_j) \leftrightsquigarrow u'_j$
(c) for every $k \in \mathcal{K}$, we have $\partial(u_k) \leftrightsquigarrow v'_k$.

Then, the following properties are satisfied:

(1) for every $i \in \mathcal{I}$, we have $\upsilon^*(w'_i) = 0$;
(2) for every $j \in \mathcal{J}$, we have $\partial(u'_j) = 0$;
(3) for every $k \in \mathcal{K}$, we have $\mathrm{e}(v'_k) = 0$.
(4) If, furthermore, out of the three following statements:
(i) the $(v_i)_{i\in\mathcal{I}}$ and $(v'_k)_{k\in\mathcal{K}}$ form a total basis of the P-part of the Witt groups of Y with support in Z, over X,
(ii) the $(w'_i)_{i\in\mathcal{I}}$ and $(w_j)_{j\in\mathcal{J}}$ form a total basis of the P-part of the Witt groups of Y, over X,
(iii) the $(u_k)_{k\in\mathcal{K}}$ and $(u'_j)_{j\in\mathcal{J}}$ form a total basis of the P_U-part of the Witt groups of U, over X,

two are true, then the remaining one is also true.

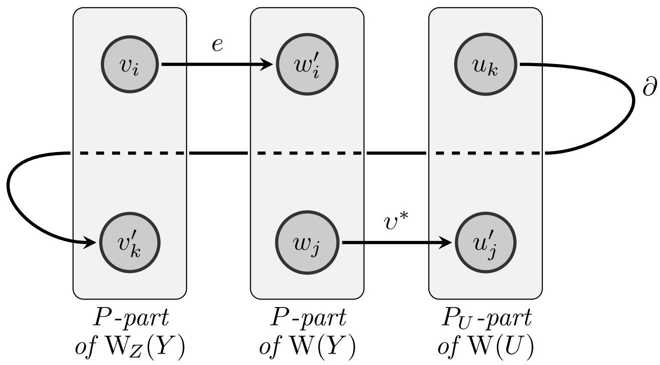

FIGURE 1. Families mapping to each other up to lax-similitude in Theorem 7.1. No arrow means mapped to zero.

Proof. Parts (1)–(3) follow immediately from Remark 2.10. The proof of (4) goes through as for classical modules over a ring. Choose a class $p \in P$ over Y, which by hypothesis is the same thing as choosing its image $f^*(p) \in f^*P$, i.e. a class in f^*P over U. Up to replacing the u'_j, v'_k and w'_i up to lax-similitude, which does not change their total-basis qualities by Corollary 6.14, we can assume that relations (a)–(c) are equalities. Now, choosing X-alignments as in Proposition 6.9 for those u_k, u'_j, v_i, v'_k, w_j and w'_i which are X-aligned with L, one can define three homomorphisms θ as in that proposition. One can construct a split exact sequence at the level of Witt groups of X, and compare it with the localization long exact sequence for Y and L via the various θ. This is a morphism of long exact sequences by the compatibilities of the lax product structure with extension of support, restriction to an open and connecting homomorphism, see Remark 5.9. The Five Lemma then gives the result. We leave the details to the reader. □

7.2. **Remark.** The same theorem holds with support, i.e. replacing $\mathrm{W}(Y)$ by $\mathrm{W}_{Z'}(Y)$, and consequently $\mathrm{W}_Z(Y)$ by $\mathrm{W}_{Z\cap Z'}(Y)$ and $\mathrm{W}(U)$ by $\mathrm{W}_{Z'\cap U}(U)$.

7.3. **Remark.** The benefit of this theorem, together with the one on dévissage, is that we can build a total basis on Y out of smaller ones on Y with support in Z and on U. As in Remark 6.17 and for the same reasons, the injectivity assumption on $P \to \mathrm{Pic}_X(U)/2$ is not really restrictive in actual computations.

7.4. **Remark.** All this "total" formalism still holds in the non-necessarily regular case with the following modifications. All schemes should be noetherian and separated. The category $\mathcal{S}_X$ should be replaced by the category of X-schemes Y that have a dualizing complex (not necessarily injectively bounded), with the conditions on Picard groups left unchanged: the important point is that two dualizing complexes always differ by tensoring by a line bundle (and a shift). The Witt groups considered for such schemes Y should be the coherent Witt groups, and the formalism will mimic how they behave as a (total) module over the *locally free* Witt groups of X. In particular, the X-alignment of definition 5.1 should be replaced by an isomorphism $\phi : M^{\otimes 2} \otimes \pi_Y^* N \otimes K' \xrightarrow{\sim} K$ where M and N are line bundles, as before, but K and K' are dualizing complexes. Pull-backs of coherent Witt groups should only be considered along flat morphisms preserving dualizing complexes (e.g. open embeddings in the localization sequence). Morphisms involving pull-backs of locally free Witt groups can be considered without restriction (e.g. pull-backs from the base X).

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